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GEOMETRY OF THE TOTAL TIME ON TEST TRANSFORM

by
RICHARD E. BARLOW

OPERATIONS RESEARCH CENTER

UNIVERSITY OF CALIFORNIA · BERKELEY

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ABSTRACT

Total time on test (TTT) plots provide a useful graphical method for tentative identification of failure distribution models. Identification is based on properties of the TTT transform. New properties of the TTT transform distribution are obtained. In particular, it is shown that a non-IFRA distribution may have an anti-starshaped transform. Hence, TTT transforms may only be useful for determining local properties of the failure rate function and not the failure rate average function.

GEOMETRY OF THE TOTAL TIME ON TEST TRANSFORM

by

Richard E. Barlow

1. INTRODUCTION

The geometry of the total time on test transform is helpful in interpreting total time on test data plots [cf. Barlow and Campo (1975)]. In particular, it is possible to infer tentative probability distribution models based on total time on test plots.

Let F be a failure distribution, i.e., $F(0^-) = 0$ and $\bar{F} = 1 - F$. Define

$$H_F^{-1}(t) = \int_0^{F^{-1}(t)} \bar{F}(x)dx \qquad 0 \leq t \leq 1 ,$$

the *total time on test transform* of F. It is easy to verify that, H_F, the inverse of H_F^{-1} is a distribution function. Also, H has support in $[0,\theta]$ if θ is the mean of F, since

$$\int_0^{F^{-1}(1)} \bar{F}(x)dx = \int_0^{\infty} xdF(x) = \theta$$

by an integration by parts. It is easy to verify that if $F(x) = 1 - e^{-x/\theta}$, then the corresponding $H_F(x) = x/\theta$ for $0 \leq x \leq \theta$. The result that our transform carries the exponential distribution into the rectangular distribution on $[0,\theta]$ is important. It is easy to verify that in this case $H_F \underset{c}{\leq} F$; i.e., $F^{-1}H_F(x)$ is convex for $0 \leq x \leq \theta$.

If $F(x) = x$ for $0 \le x \le 1$; i.e., F is uniform on $[0,1]$, then $H_F(x) = 1 - \sqrt{1 - 2x}$ for $0 \le x \le \frac{1}{2}$, since F has mean $\frac{1}{2}$ in this case. Again $H_F \underset{c}{<} F$. In general $H_F^{-1}F(x) = \int_0^x \bar{F}(u)du$ is concave since $\bar{F}$ is nonincreasing. Hence, the inverse $F^{-1}H_F(x)$ is convex for $0 \le x \le \theta$ and

$$H_F \underset{c}{<} F$$

for all F with $F(0^-) = 0$.

As was proved in Barlow, Bartholomew, Bremner and Brunk (1972), total time on test data plots tend to the total time on test transform of the underlying failure distribution as the sample size tends to infinity. In order to interpret total time on test data plots, we need to understand the relationship between F and its transform. The following table summarizes the connections.

Life Distribution F		Hazard Function $R = \log \bar{F}$		Total Time on Test Transform Distribution H_F
Exponential	$\Leftrightarrow$	linear	$\Leftrightarrow$	linear
IFR	$\Leftrightarrow$	convex	$\Leftrightarrow$	convex
DFR	$\Leftrightarrow$	concave	$\Leftrightarrow$	concave
IFRA	$\Leftrightarrow$	starshaped	$\Rightarrow$	starshaped
DFRA	$\Leftrightarrow$	anti-starshaped	$\Rightarrow$	anti-starshaped

TABLE 1

Logical Connections Between Life Distributions,

Hazard Functions and TTT Transform Distributions

A function g defined on $[0,b)$ such that $\frac{g(x)}{x}$ is nondecreasing on $[0,b)$ is said to be starshaped with respect to the origin. If $G(x) = 1 - e^{-x}$, then F is IFRA (for increasing failure rate average if and only if $\frac{G^{-1}F(x)}{x}$ is nondecreasing for $0 \leq x \leq F^{-1}(1)$. The function $G^{-1}F(x) = R(x)$ is said to be starshaped with respect to the origin. As the last two implications indicate, IFRA and DFRA distribution families are *not* characterized by corresponding properties of the TTT transform distribution. However IFR and DFR distribution families are characterized by corresponding properties of the TTT transform distribution.

To verify the implications in the table for the IFR (DFR) case, first assume F absolutely continuous with failure rate function, r. If F is IFR (DFR), then

$$\frac{d}{dt} H_F^{-1}(t)\Big|_{t=F(x)} = \frac{1}{r(x)}$$

is decreasing (increasing) in x which implies H_F^{-1} is concave (convex); i.e., H_F is convex (concave). Conversely, if H_F^{-1} is concave (convex), the failure rate function is increasing (decreasing). To see this, note that every IFR (DFR) distribution can be approximated arbitrarily closely by an absolutely continuous IFR (DFR) distribution. Since the limit of a sequence of concave (convex) transforms is concave (convex) on $[0,1]$, it follows that F is IFR (DFR) if and only if H_F^{-1} is concave (convex).

The IFRA distributions govern the lifelength of coherent systems with statistically independent components whose life distributions are IFR (or, more generally, IFRA). [Birnbaum, Esary and Marshall (1966) or Barlow and Proschan (1975)]. They also arise in other reliability contexts.

For these reasons, we are interested in the transforms of IFRA distributions. In the next section we show that if F is IFRA, then its transform distribution, H_F, is starshaped; i.e., $\frac{H_F(x)}{x}$ is nondecreasing in $0 \leq x \leq \theta$. Unfortunately, the converse is not true.

2. PRESERVATION OF PARTIAL ORDERINGS ON CLASSES OF FAILURE DISTRIBUTIONS

Let $R(x) = -\log \bar{F}(x)$ be the hazard function of F as before and let $G(x) = 1 - e^{-x}$. Observe that $G^{-1}F(x) = R(x)$ so that if F is IFR, $G^{-1}F(x)$ is convex on the support of F and conversely. If F is IFRA, $\frac{G^{-1}F(x)}{x}$ is nondecreasing in $x \geq 0$ and conversely. This leads to a partial ordering on the space of failure distributions which we call "star ordering." Let $\mathcal{F}$ be the class of continuous distributions on $[0,\infty)$ and $\{\text{deg.}\}$, the class of degenerate distributions.

Definition:

$F_1 \underset{*}{<} F_2$ $\Big($i.e., F_1 is star ordered with respect to F_2 if F_1, $F_2 \in \mathcal{F} \cup \{\text{deg.}\}$ and $\frac{F_2^{-1}F_1(x)}{x}$ is nondecreasing in x for $0 \leq x \leq F_1^{-1}(1)\Big)$.

According to this definition, every distribution in $\mathcal{F}$ is star ordered with respect to a degenerate distribution. Let $F_\alpha(x) = 1 - e^{-x^\alpha}$ for $x \geq 0$ and $\alpha > 0$. It is easy to show that if $0 < \alpha_1 < \alpha_2$, then $F_{\alpha_2} \underset{*}{<} F_{\alpha_1}$. Since F_α has failure rate $\alpha x^{\alpha-1}$, it is clear that the failure rate of F_{α_2} is "increasing faster" than the failure rate of F_{α_1}. If $0 < \alpha < 1$, F_α is DFR. If $\alpha > 1$, F_α is IFR and F_1 is exponential.

Definition:

$F_1 \underset{c}{<} F_2$ $\Big($i.e., F_1 is convex ordered with respect to F_2 if F_1, $F_2 \in \mathcal{F} \cup \{\text{deg.}\}$ and $F_2^{-1}F_1(x)$ is convex in x for $0 \leq x \leq F_1^{-1}(1)\Big)$.

It is not hard to show that c-ordering implies star ordering, but not conversely. Our main theorem is that the TTT transform distribution preserves both orderings.

Theorem 2.1:

Let F_1 , $F_2 \in \mathcal{F}$.

(a) If $F_1 \underset{c}{\leq} F_2$, then $H_{F_1} \underset{c}{\leq} H_{F_2}$.

(b) If $F_1 \underset{*}{\leq} F_2$, then $H_{F_1} \underset{*}{\leq} H_{F_2}$.

The following corollary provides the primary application of the theorem.

Corollary 2.2:

If F_1 , $F_2 \in \mathcal{F}$ and $F_1 \underset{*}{\leq} F_2$, then

(a) $\dfrac{H_{F_2}^{-1}(t)}{H_{F_1}^{-1}(t)}$ is nondecreasing in $0 \leq t \leq 1$;

(b) $\dfrac{H_{F_1}^{-1}(t)}{H_{F_1}^{-1}(1)} \geq \dfrac{H_{F_2}^{-1}(t)}{H_{F_2}^{-1}(1)}$ for $0 \leq t \leq 1$.

Proof of Corollary:

By Theorem 2.1, Part (b), $H_{F_1} \underset{*}{\leq} H_{F_2}$ so $\dfrac{H_{F_2}^{-1} H_{F_1}(x)}{x}$ is nondecreasing in $0 \leq x \leq F_1^{-1}(1)$. Let $t = H_{F_1}(x)$ and Part (a) of the corollary is immediate. (b) is a trivial consequence of (a).||

Figures 1 and 2 are graphical plots of the scaled transforms of gamma and Weibull distributions. They visually confirm Part (b) of the corollary.

Scaled Total Time on Test Transforms

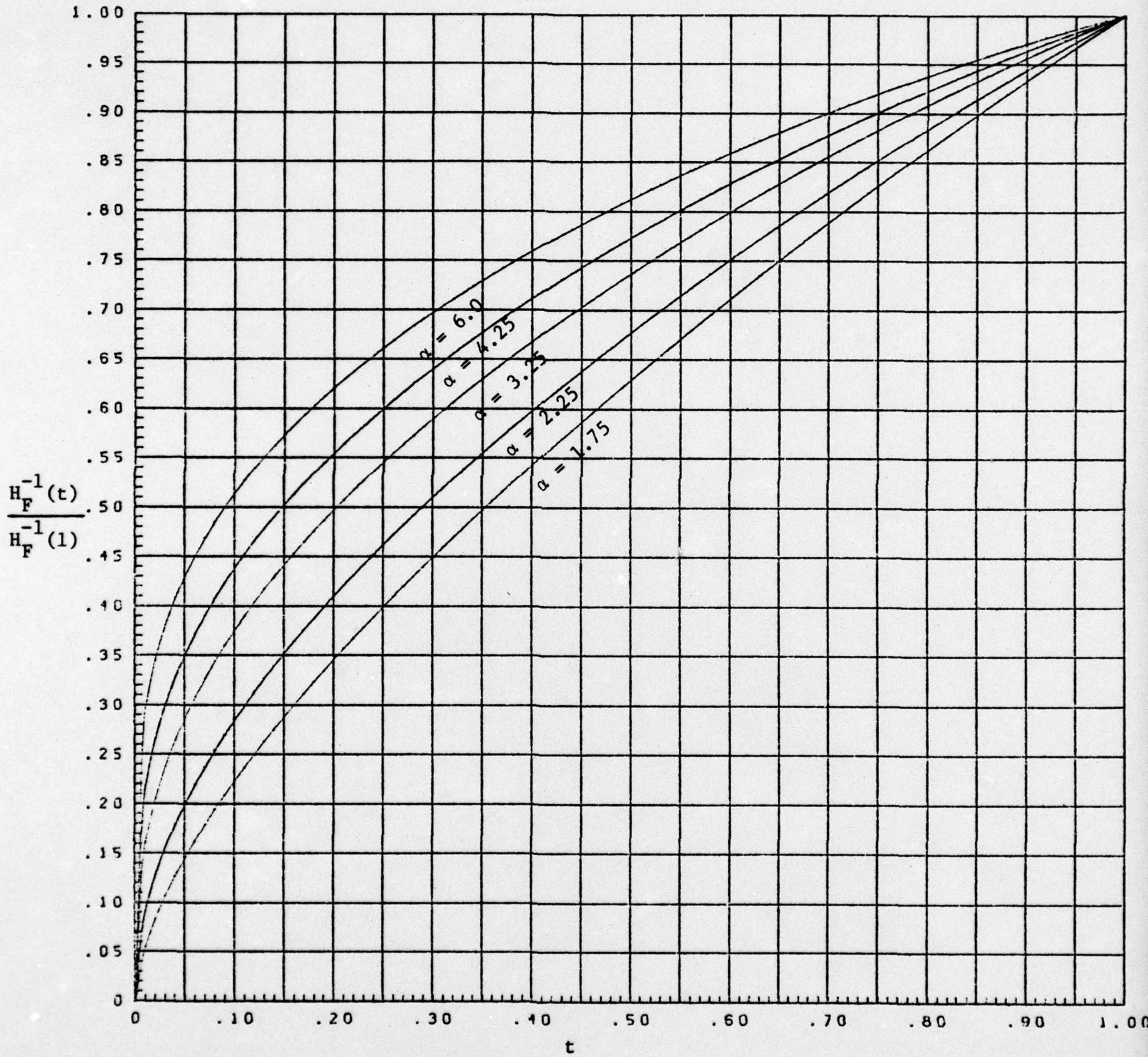

FIGURE 1

GAMMA DISTRIBUTION

(shape parameter α)

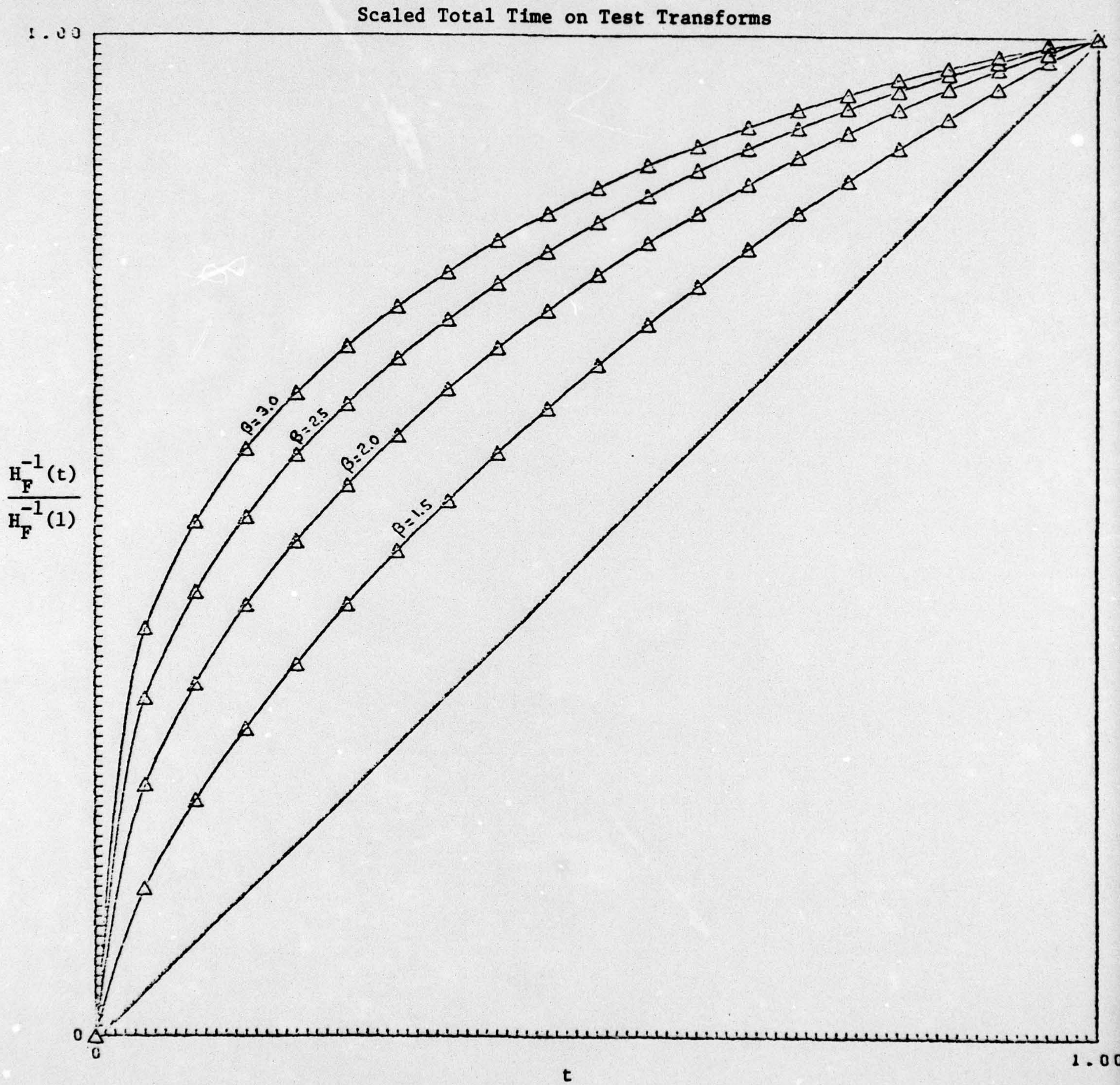

FIGURE 2

WEIBULL DISTRIBUTION

(shape parameter β)

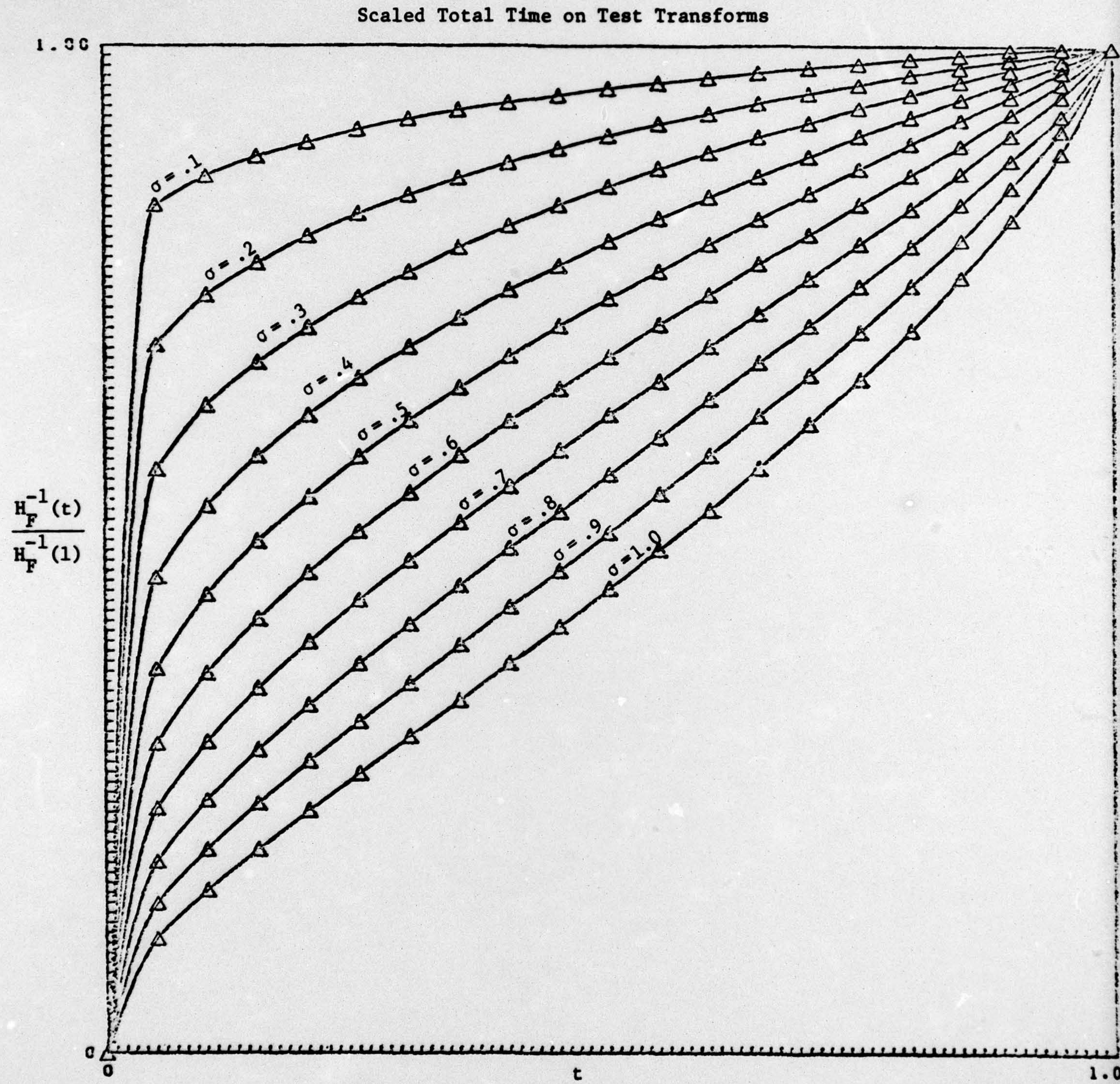

FIGURE 3

LOGNORMAL DISTRIBUTION

(shape parameter σ)

<u>Proof of Theorem 2.1</u>:

(a) Assume $F_1 \underset{c}{<} F_2$. We wish to show $H_{F_2}^{-1} H_{F_1}(x)$ is convex in $0 \le x \le F_1^{-1}(1)$. First assume F_1 and F_2 are absolutely continuous. Then we need only show $\frac{d}{dx} H_{F_2}^{-1}\left[H_{F_1}(x)\right] = \frac{d}{dx} \int_0^{F_2^{-1}\left[H_{F_1}(x)\right]} \bar{F}_2(u)du$ is nondecreasing in $0 \le x \le F_1^{-1}(1)$. Now

$$\frac{d}{dx} \int_0^{F_2^{-1}\left[H_{F_1}(x)\right]} \bar{F}_2(u)du = \left[\frac{1 - H_{F_1}(x)}{f_2\left[F_2^{-1} H_{F_1}(x)\right]}\right] \frac{d}{dx} H_{F_1}(x) .$$

Let $x = H_{F_1}^{-1}(t)$ so that $\frac{dx}{dt} = \frac{1 - t}{f_1\left[F_1^{-1}(t)\right]}$ and

$$\frac{dt}{dx} = \frac{f_1\left[F_1^{-1}(t)\right]}{1 - t}\Bigg|_{t=H_{F_1}(x)} = \frac{f_1\left[F_1^{-1} H_{F_1}(x)\right]}{1 - H_{F_1}(x)} .$$

Hence $\frac{dH_{F_1}(x)}{dx} = \frac{dt}{dx} = \frac{f_1\left[F_1^{-1} H_{F_1}(x)\right]}{1 - H_{F_1}(x)}$ and $\frac{d}{dx} \int_0^{F_2^{-1}\left[H_{F_1}(x)\right]} \bar{F}_2(u)du = \frac{f_1\left[F_1^{-1} H_{F_1}(x)\right]}{f_2\left[F_2^{-1} H_{F_1}(x)\right]}$.

But $F_1 \underset{c}{<} F_2$ implies

$$\frac{d}{dx} F_2^{-1} F_1(x) = \frac{f_1(x)}{f_2\left[F_2^{-1} F_1(x)\right]}$$

is nondecreasing in $0 \leq x \leq F_1^{-1}(1)$. Since $F_1^{-1}H_{F_1}(x) = t$ is nondecreasing in $0 \leq x \leq F_1^{-1}(1)$, a change of variable completes the argument.

Since continuous distributions can be approximated arbitrarily closely by absolutely continuous distributions, the proof of part (a) is complete.

To prove (b) we will need the following fundamental lemma.

2.3: Fundamental Lemma

If $R(0) = 0$, $\frac{R(x)}{x}$ is nondecreasing in $x \geq 0$ and $0 \leq N(x) \leq \frac{1}{x}\int_0^x N(u)du$, then

$$\frac{\int_0^x N(u)dR(u)}{\int_0^x N(u)du} \tag{2.1}$$

is nondecreasing in $x \geq 0$. [Note that if $N(x)$ is nonincreasing, the assumption on $N(x)$ is automatically satisfied].

Proof:

R can be approximated arbitrarily closely from below by positive linear combinations of simple functions of the form

$$R(x) = \begin{cases} 0 & x < x_0 \\ x & x \geq x_0 . \end{cases}$$

Hence we need only verify the lemma for simple functions. The general result follows from the Lebesgue monotone convergence theorem. For a simple function, R

$$\frac{\int_0^x N(u)dR(u)}{\int_0^x N(u)du} = \begin{cases} 0 & x < x_0 \\ \dfrac{x_0 N(x_0) + \int_{x_0}^x N(u)du}{\int_0^x N(u)du} & x \geq x_0 \end{cases} .$$

Hence, for $x \geq x_0$

$$\frac{\int_0^x N(u)dR(u)}{\int_0^x N(u)du} = 1 + \frac{\left[x_0 N(x_0) - \int_0^{x_0} N(u)du\right]}{\int_0^x N(u)du} .$$

By assumption, $N(x_0) \leq \frac{1}{x_0} \int_0^{x_0} N(u)du$ so that the lemma follows.||

<u>Theorem 2.1, Part (b)</u>:

Let $R(x) = F_2^{-1}F_1(x)$. By assumption, $\frac{R(x)}{x}$ is nondecreasing in $0 \leq x \leq F_1^{-1}(1)$. Let $N(u) = \bar{F}_1(u)$, $x = F_1^{-1}(t)$ and substitute in (2.1) to obtain

$$\frac{\int_0^{F_1^{-1}(t)} \bar{F}_1(u)\,dR(u)}{\int_0^{F_1^{-1}(t)} \bar{F}_1(u)\,du} .$$

Let $v = F_2^{-1}F_1(u) = R(u)$ so that the numerator becomes $\int_0^{F_1^{-1}(t)} \bar{F}_1(u)\,dR(u) = \int_0^{F_2^{-1}(t)} \bar{F}_2(v)\,dv$. It follows from the Fundamental Lemma that

$$\frac{H_{F_2}^{-1}(t)}{H_{F_1}^{-1}(t)} = \frac{\int_0^{F_2^{-1}(t)} \bar{F}_2(v)\,dv}{\int_0^{F_1^{-1}(t)} \bar{F}_1(u)\,du}$$

is nondecreasing in $0 \leq t \leq 1$ or $\dfrac{H_{F_2}^{-1}H_{F_1}(x)}{x}$ is nondecreasing in $0 \leq x \leq F_1^{-1}(1)$, i.e.,

$$H_{F_1} \underset{*}{<} H_{F_2} . ||$$

$H_{F_1} \underset{*}{<} H_{F_2}$ DOES NOT IMPLY $F_1 \underset{*}{<} F_2$

Let $G(x) = 1 - e^{-x}$ so that $H_G(x) = x$ for $0 \leq x \leq 1$. It is easy to find examples such that $H_F \underset{*}{<} H_G$ but $F \underset{*}{\not<} G$; i.e., F is not IFRA. Note that for $0 < t_1 < 1$, $c = -\log(1 - t_1) - t_1 > 0$. Hence

$$\bar{F}(x) = \begin{cases} 1 & 0 \le x < t_1 \\ e^{-(c+x)} & x \ge t_1 \end{cases}$$

is not IFRA since

$$\frac{R(x)}{x} = \begin{cases} 0 & 0 \le x < t_1 \\ \frac{c}{x} + 1 & x \ge t_1 \end{cases}$$

is decreasing for $x \ge t_1$.

But

$$H_F^{-1}(t) = \begin{cases} t_1 & 0 \le t < t_1 \\ t & t_1 \le t \le 1 , \end{cases}$$

is anti-starshaped; i.e., $\dfrac{H_F^{-1}(t)}{t}$ is nonincreasing in $0 < t < 1$ so that $H_F \underset{*}{<} H_G$.

The significance of this example is that an anti-starshaped total time on test plot is *not* necessarily evidence that F is IFRA.

3. A MEASURE OF IFRness

Figures 1 and 2 show scaled total time on test transformations for various parametric families of failure distributions. In each case a single shape parameter provides a measure of departure from exponentiality.

By Part (b) of Corollary 2.2, the area $\int_0^1 H_F^{-1}(u)du$ could also provide a measure of IFRness since if $F_1 \underset{c}{<} F_2$ and $\int_0^\infty x dF_1(x) = \int_0^\infty x dF_2(x)$, then

$$\int_0^1 H_{F_1}^{-1}(u)du \geq \int_0^1 H_{F_2}^{-1}(u)du \ .$$

If F has mean θ, then $\int_0^1 H_F^{-1}(u)du = \int_0^\theta x dH_F(x)$, so that the mean of H_F, the inverse of the transform of F provides a measure of the IFRness of F. The following lemma provides an easy means for calculating $\int_0^\theta x dH_F(x)$.

Lemma:

If $\int_0^\infty x dF(x) < \infty$, then

$$\int_0^\infty x dH_F(x) = 2\int_0^\infty x[1 - F(x)]dF(x) \ .$$

Proof:

Since $\int_0^\infty x dH_F(x) = \int_0^1 H_F^{-1}(u)du$, we integrate the latter by parts to obtain

$$\int_0^1 H_F^{-1}(u)du = \int_0^1 F^{-1}(u)du - \int_0^1 t(1 - t)dF^{-1}(t) .$$

Integrate by parts again to obtain

$$-\int_0^1 t(1 - t)dF^{-1}(t) = \int_0^1 F^{-1}(u)(1 - 2u)du$$

so that

$$\int_0^1 H_F^{-1}(t)dt = 2\int_0^1 (1 - u)F^{-1}(u)du = 2\int_0^\infty x[1 - F(x)]dF(x)$$

by a change of variable.||

Examples:

For $F(x) = 1 - e^{-(\lambda x)^\alpha}$ with mean, say θ ,

$$\frac{1}{\theta}\int_0^\infty x dH_F(x) = 1/2^{1/\alpha} .$$

For the gamma distribution

$$F(x) = \int_0^x \frac{\lambda^k u^{k-1} e^{-\lambda u}}{(k - 1)!} du , \qquad k = 1,2, \ldots$$

with mean $\theta = \frac{k}{\lambda}$,

$$\frac{1}{\theta}\int_0^\infty x dH_F(x) = \sum_{i=0}^{k-1} \binom{i+k}{k} \frac{1}{2^{i+k}} .$$

The numerical relationship between $\frac{1}{\theta}\int_0^\infty x dH_F(x)$ and the shape parameter for Weibull and gamma distributions is shown in the following table.

$\frac{1}{\theta}\int_0^\infty x\,dH_F(x)$	Weibull α	Gamma k
.20	.43	
.25	.50	
.30	.58	
.35	.66	
.40	.76	
.45	.87	
.50	1	1
.55	1.16	
.60	1.36	
.63	1.50	2
.65	1.61	
.69	1.87	3
.70	1.94	
.73	2.20	4
.75	2.41	5
.80	3.11	
.85	4.27	
.90	6.58	
.95	13.51	

TABLE 3.1

RELATIONSHIP BETWEEN MEASURES OF "IFRness"

REFERENCES

Barlow, R. E. and R. Campo, "Total Time on Test Processes and Applications to Failure Data Analysis," in RELIABILITY AND FAULT TREE ANALYSIS, edited by R. E. Barlow, J. Fussell and N. Singpurwalla. Conference volume published by SIAM, Philadelphia, 1975.

Barlow, R. E., D. J. Bartholomew, J. M. Bremner and H. D. Brunk, STATISTICAL INFERENCE UNDER ORDER RESTRICTIONS, John Wiley and Sons, 1972.

Birnbaum, Z. W., J. Esary and A. W. Marshall, "Stochastic Characterization of Wearout for Components and Systems," Ann. Math. Statist., Vol. 37, pp. 816-825.

Barlow, R. E. and F. Proschan, STATISTICAL THEORY OF RELIABILITY AND LIFE TESTING, Holt, Rinehart and Winston, 1975.